Topological applications of Wadge theory I

Andrea Medini

Kurt Gödel Research Center
University of Vienna

January 26, 2020

Everybody loves

homogeneous stuff!

Topological homogeneity

A space is homogeneous if all points “look alike” from a global point of view:

Definition

A space X is *homogeneous* if for every $x, y \in X$ there exists a homeomorphism $h : X \longrightarrow X$ such that $h(x) = y$.

Non-examples:

- $\omega + 1$ (Because of the limit point)
- $[0, 1]^n$ whenever $1 \leq n < \omega$
  (Points on the boundary are different from points in the interior)
- The Stone-Čech remainder $\omega^* = \beta\omega \setminus \omega$
  (W. Rudin, 1956, under CH, because of P-points)
  (Frolík, 1967, using a cardinality argument)
  (Kunen, 1978, by proving the existence of weak P-points)

Examples:

- Any topological group
- Any product of homogeneous spaces
- Any open subspace of a zero-dimensional homogeneous space
- The Hilbert cube $[0,1]^\omega$ (Keller, 1931)
- X^ω for every zero-dimensional first-countable X
  (Dow and Pearl, 1997, based on work of Lawrence)

Homogeneous spaces are decently understood.
Compact homogeneous spaces are shrouded in mystery:

Question (Van Douwen, 1970s)

Is there a compact homogeneous space with more than $\mathfrak{c}$ pairwise disjoint non-empty open sets?

Question (W. Rudin, 1958)

Is there a compact homogeneous space with no non-trivial convergent ω-sequences?

Strong homogeneity

Definition

A space X is *strongly homogeneous* (or *h-homogeneous*) if every non-empty clopen subspace of X is homeomorphic to X.

Examples:

- Any connected space
- $\mathbb{Q}$, 2^ω, ω^ω (Use their characterizations)
- Any product of zero-dimensional strongly homogeneous spaces (Medini, 2011, building on work of Terada, 1993)
- Erdős space $\mathfrak{E} = \{x \in \ell^2 : x_n \in \mathbb{Q} \text{ for all } n \in \omega\}$ (Dijkstra and van Mill, 2010)

Non-examples:

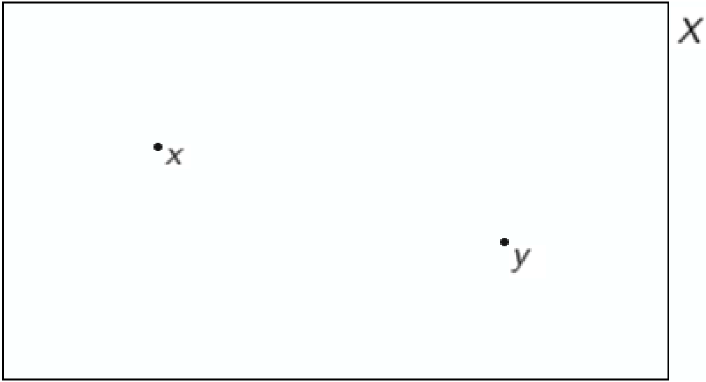
- Discrete spaces with at least two elements
- $\omega \times 2^\omega$

Is "strong" a good choice of word?

Not particularly. For example, ω^* is strongly homogeneous but not homogeneous. Things get better under additional assumptions:

Theorem (folklore)

Let X be a first-countable zero-dimensional space. If X is strongly homogeneous then X is homogeneous.



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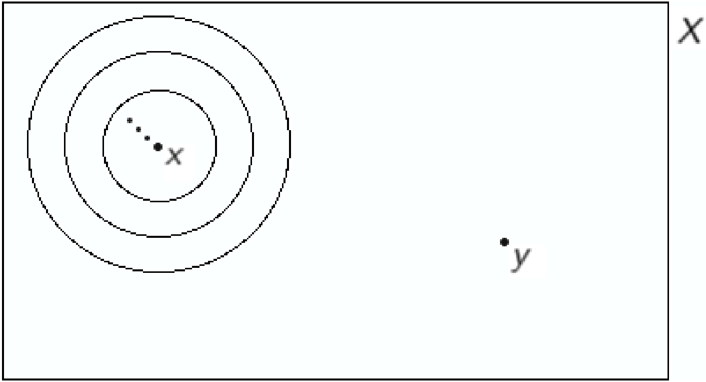

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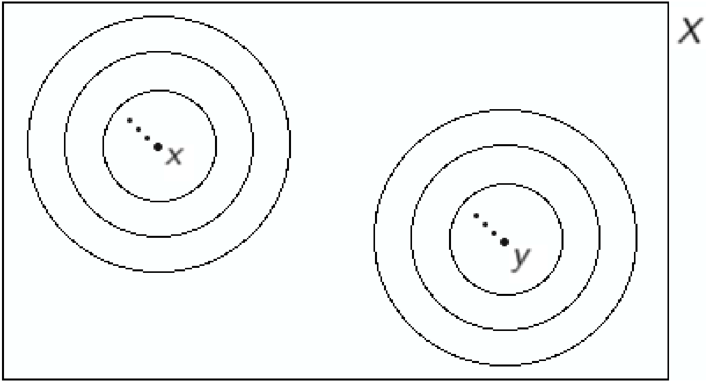

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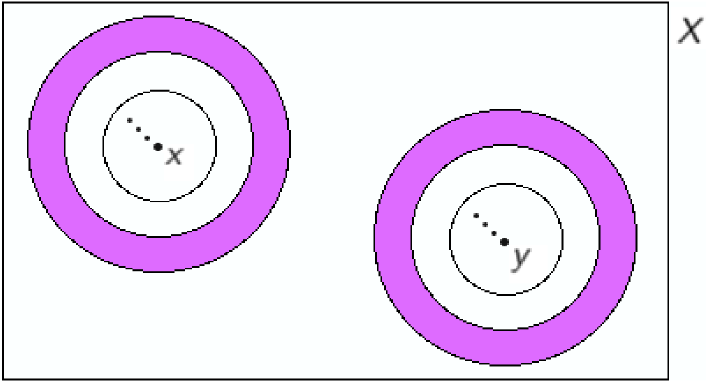

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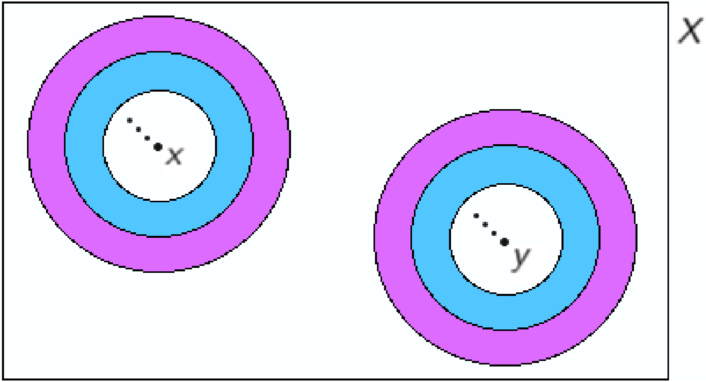


The locally compact case (the trivial case)

From now on, all spaces are separable and metrizable.

Proposition

Let X be a locally compact zero-dimensional space. Then the following conditions are equivalent:

- *X is homogeneous*
- *X is discrete, $X \approx \omega \times 2^\omega$, or $X \approx 2^\omega$*

Two open questions

Question (Terada, 1993)

Is X^ω strongly homogeneous for every zero-dimensional space X?

Question (Medvedev, 2012)

Is X strongly homogeneous for every meager zero-dimensional homogeneous space X?

An example of van Douwen

Theorem (van Douwen, 1984)

There exists a subspace X of $\mathbb{R}$ with the following properties:

- *X is a Bernstein subset of $\mathbb{R}$*
- *X is a subgroup of $(\mathbb{R}, +)$*
- *There exists a measure μ on the Borel subsets of X such that $A \approx B$ implies $\mu(A) = \mu(B)$ whenever $A, B \subseteq X$ are Borel*

Given a Borel subset A of X, the measure of A is defined by:

$$\mu(A) = \text{Lebesgue measure of } \widetilde{A}$$

where $\widetilde{A}$ is a Borel subset of $\mathbb{R}$ such that $\widetilde{A} \cap X = A$.

Corollary

There exists a zero-dimensional homogeneous space that is not locally compact space and not strongly homogeneous.

The main result

In his remarkable Ph.D. thesis, van Engelen obtained a complete classification of the zero-dimensional homogeneous Borel spaces. As a corollary, he proved the following:

Theorem (van Engelen, 1986)

Let X be a zero-dimensional Borel space that is not locally compact. If X is homogeneous then X is strongly homogeneous.

Can the “Borel” assumption be dropped? Certainly not in ZFC, by van Douwen’s example. However:

Theorem (Carroy, Medini, Müller)

Work in $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$. *Let X be a zero-dimensional space that is not locally compact. If X is homogeneous then X is strongly homogeneous.*

Notice that the above result also gives consistent “yes” answers to both Terada’s and Medvedev’s questions. It is still open whether AD is really needed for answering those questions.

A few words about the axioms

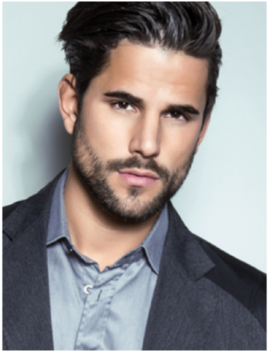
AD denotes the Axiom of Determinacy: every game on ω is determined (either Player I or Player II has a winning strategy).
DC denotes the principle of Dependent Choices: it is strictly intermediate in strength between the Axiom of Choice and the Countable Axiom of Choice.

1. The set-theoretic universe is extremely regular under AD: every set of reals has the Baire property, the perfect set property, and is Lebesgue-measurable
2. AD is incompatible with the Axiom of Choice
3. ZF + DC + AD is consistent (assuming large cardinals)
4. ZF + DC is sufficient to carry out recursions of length ω
5. DC is equivalent to Baire's Category Theorem (Blair, 1977)
6. ZF + DC proves Borel Determinacy (Martin, 1975)

Up until our main result, our ambient theory was ZFC. From now on, we will always be working in ZF + DC.

The Wadge Brigade

Raphaël Carroy



Andrea Medini

Sandra Müller

The Wadge Brigade (for real)

Raphaël Carroy

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Wadge theory: basic definitions

Let Z be a set and $\mathbf{\Gamma} \subseteq \mathcal{P}(Z)$. Define $\check{\mathbf{\Gamma}} = \{Z \setminus A : A \in \mathbf{\Gamma}\}$. We say that $\mathbf{\Gamma}$ is *selfdual* if $\mathbf{\Gamma} = \check{\mathbf{\Gamma}}$. Also define $\Delta(\mathbf{\Gamma}) = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$.

Definition (Wadge, 1984)

Let Z be a space. Given $A, B \subseteq Z$, we will write $A \leq B$ if there exists a continuous function $f : Z \longrightarrow Z$ such that $A = f^{-1}[B]$. In this case, we will say that A is *Wadge-reducible* to B, and that f *witnesses* the reduction.

Definition (Wadge, 1984)

Let Z be a space. Given $A \subseteq Z$, define

$$[A] = \{B \subseteq Z : B \leq A\}$$

We will say that $\mathbf{\Gamma} \subseteq \mathcal{P}(Z)$ is a *Wadge class* if there exists $A \subseteq Z$ such that $\mathbf{\Gamma} = [A]$. The set A is *selfdual* if $[A]$ is selfdual.

First examples of Wadge classes

From now on, unless we specify otherwise, we will always assume that Z is an uncountable zero-dimensional Polish space.

- $\{\varnothing\}$ and $\{Z\}$ (These are the minimal ones)
- $\mathbf{\Delta}^0_1(Z)$ is their immediate successor
  (Generated by an arbitrary proper clopen set)

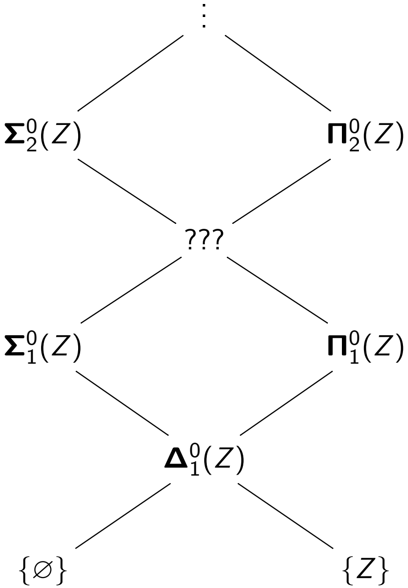
Let $1 \leq \xi < \omega_1$. Recall that $\mathbf{\Sigma}^0_\xi(Z)$ has a 2^ω-universal set U. This means that $U \in \mathbf{\Sigma}^0_\xi(2^\omega \times Z)$ and

$$\mathbf{\Sigma}^0_\xi(Z) = \{U_x : x \in 2^\omega\}$$

where $U_x = \{y \in Z : (x, y) \in U\}$ denotes the vertical section.

- $\mathbf{\Sigma}^0_\xi(Z)$ and $\mathbf{\Pi}^0_\xi(Z)$ (Generated by a universal set)
- $\mathbf{\Sigma}^1_n(Z)$ and $\mathbf{\Pi}^1_n(Z)$ for $n \geq 1$ (As above)

The Wadge hierachy: a first glimpse



Why do we need determinacy?

Lemma (Wadge, 1984)

Assume AD. *Let* $A, B \subseteq Z$. *Then either* $A \leq B$ *or* $B \leq Z \setminus A$.

Here are two simple (but very useful) applications:

- In the poset $\mathbb{W}(Z)$ of all Wadge classes in Z ordered by $\subseteq$, antichains have size at most 2
- If $\boldsymbol{\Gamma}$ is a Wadge class and $A \in \boldsymbol{\Gamma} \setminus \check{\boldsymbol{\Gamma}}$ then $[A] = \boldsymbol{\Gamma}$

Theorem (Martin, Monk)

Assume AD. *The poset* $\mathbb{W}(Z)$ *is well-founded.*

This yields the definition of Wadge rank.

- By the two results above, $\mathbb{W}(Z)$ becomes a well-order if we identify every Wadge class $\boldsymbol{\Gamma}$ with its dual class $\check{\boldsymbol{\Gamma}}$
- The length of this well-order is Θ

From now on, we will always assume that AD holds.

Playing around with partitioned unions

Definition

Given $\xi < \omega_1$, define $\mathrm{PU}_\xi(\boldsymbol{\Gamma})$ as the collection of all sets of the form

$$\bigcup_{n\in\omega}(A_n \cap V_n)$$

where $A_n \in \boldsymbol{\Gamma}$ for $n \in \omega$ and $\{V_n : n \in \omega\} \subseteq \boldsymbol{\Delta}^0_{1+\xi}(Z)$ is a partition of Z. A set in this form is called a *partitioned union* of sets in $\boldsymbol{\Gamma}$.

- $\mathrm{PU}_0(\boldsymbol{\Gamma}) = \boldsymbol{\Gamma}$ whenever $\boldsymbol{\Gamma}$ is a Wadge class
- $\mathrm{PU}_0(\boldsymbol{\Delta})$ is selfdual whenever $\boldsymbol{\Delta}$ is selfdual
- If $\boldsymbol{\Gamma}$ a non-selfdual Wadge class, then $\mathrm{PU}_0(\boldsymbol{\Gamma} \cup \check{\boldsymbol{\Gamma}})$ is a Wadge class (the immediate successor of $\boldsymbol{\Gamma}$)
- If Z is non-compact and $\langle \boldsymbol{\Gamma}_n : n \in \omega\rangle$ is a strictly increasing sequence of non-selfdual Wadge classes, then $\mathrm{PU}_0(\bigcup_{n\in\omega} \boldsymbol{\Gamma}_n)$ is a Wadge class (the supremum of $\{\boldsymbol{\Gamma}_n : n \in \omega\}$)

The analysis of selfdual sets

The following fundamental result reduces the study of self-dual Wadge classes to the study of non-selfdual Wadge classes:

Theorem (see Motto Ros, 2009)

Let $\mathbf{\Delta}$ be a selfdual Wadge class in Z. Then there exist non-selfdual Wadge classes $\mathbf{\Gamma}_n$ for $n \in \omega$ such that

$$\mathbf{\Delta} = \mathrm{PU}_0\left(\bigcup_{n\in\omega}(\mathbf{\Gamma}_n \cup \check{\mathbf{\Gamma}}_n)\right)$$

- The above result shows that Wadge classes whose rank has uncountable cofinality can never be selfdual
- By the same argument, if Z is compact, this is the case for all Wadge classes whose rank is a limit ordinal
- However, it can be shown that the poset of non-selfdual Wadge classes does not depend on the ambient space

Hausdorff operations

Definition (Hausdorff, 1927)

Given $D \subseteq \mathcal{P}(\omega)$, define

$$\mathcal{H}_D(A_0, A_1, \ldots) = \{x \in Z : \{n \in \omega : x \in A_n\} \in D\}$$

whenever $A_0, A_1, \ldots \subseteq Z$. We will call functions of this form *Hausdorff operations* (or *ω-ary Boolean operations*).

Given $n \in \omega$, set $S_n = \{A \subseteq \omega : n \in A\}$. Then:

- $\mathcal{H}_{S_n}(A_0, A_1, \ldots) = A_n$
- $\bigcap_{i \in I} \mathcal{H}_{D_i}(A_0, A_1, \ldots) = \mathcal{H}_D(A_0, A_1, \ldots)$, where $D = \bigcap_{i \in I} D_i$
- $\bigcup_{i \in I} \mathcal{H}_{D_i}(A_0, A_1, \ldots) = \mathcal{H}_D(A_0, A_1, \ldots)$, where $D = \bigcup_{i \in I} D_i$
- $Z \setminus \mathcal{H}_D(A_0, A_1, \ldots) = \mathcal{H}_{\mathcal{P}(\omega) \setminus D}(A_0, A_1, \ldots)$ for all $D \subseteq \mathcal{P}(\omega)$

Hence, any operation obtained by combining unions, intersections and complements can be expressed as a Hausdorff operation.

The difference hierarchy

Given $1 \leq \eta < \omega_1$, define the Hausdorff operation D_η as follows:

- $\mathrm{D}_1(A_0) = A_0$
- $\mathrm{D}_2(A_0, A_1) = A_1 \setminus A_0$
- $\mathrm{D}_3(A_0, A_1, A_2) = A_0 \cup (A_2 \setminus A_1)$

$$\vdots$$

- $\mathrm{D}_\omega(A_0, A_1, \ldots) = (A_1 \setminus A_0) \cup (A_3 \setminus A_2) \cup \cdots$
- $\mathrm{D}_{\omega+1}(A_0, A_1, \ldots, A_\omega) = A_0 \cup (A_2 \setminus A_1) \cup \cdots \cup (A_\omega \setminus \bigcup_{n<\omega} A_n)$

$$\vdots$$

Given $1 \leq \xi < \omega_1$, define:

$$\mathrm{D}_\eta(\boldsymbol{\Sigma}^0_\xi) = \{\mathrm{D}_\eta(A_\mu : \mu < \eta) : \text{each } A_\mu \in \boldsymbol{\Sigma}^0_\xi \text{ and } (A_\mu : \mu < \eta) \text{ is increasing}\}$$

It can be shown that $\mathrm{D}_\eta(\boldsymbol{\Sigma}^0_\xi) \subsetneq \mathrm{D}_\mu(\boldsymbol{\Sigma}^0_\xi)$ whenever $\eta < \mu$.

Wadge classes from Hausdorff operations

Definition

Given $D \subseteq \mathcal{P}(\omega)$, define

$$\mathbf{\Gamma}_D(Z) = \{\mathcal{H}_D(A_0, A_1, \dots) : A_0, A_1, \dots \in \mathbf{\Sigma}^0_1(Z)\}$$

By fixing a 2^ω-universal set for $\mathbf{\Sigma}^0_1(Z)$ and "applying $\mathcal{H}_D$ to it", one obtains the following:

Theorem (Addison for $Z = \omega^\omega$)

Let $D \subseteq \mathcal{P}(\omega)$. Then $\mathbf{\Gamma}_D(Z)$ is a non-selfdual Wadge class.

In particular, each $\mathrm{D}_\eta(\mathbf{\Sigma}^0_1(Z))$ is a non-selfdual Wadge class.
In fact, it can be shown that they and their duals exhaust the non-selfdual Wadge classes contained in $\mathbf{\Delta}^0_2(Z)$.
The analog statement for $\mathbf{\Delta}^0_3(Z)$ is false! However:

Theorem (Hausdorff and Kuratowski)

$\mathbf{\Delta}^0_{\xi+1}(Z) = \bigcup_{1 \leq \eta < \omega_1} \mathrm{D}_\eta(\mathbf{\Sigma}^0_\xi(Z))$

Relativization: yet another reason to love Hausdorff operations

When one tries to give a systematic exposition of Wadge theory, it soon becomes apparent that it would be very useful to be able to say when A and B belong to "the same" Wadge class $\mathbf{\Gamma}$, even when $A \subseteq Z$ and $B \subseteq W$ for distinct ambient spaces Z and W. (This is clear in some particular cases, like $\mathbf{\Gamma} = \mathbf{\Pi}^0_2$ or $\mathbf{\Gamma} = \mathrm{D}_5(\mathbf{\Sigma}^0_1)$, but what about arbitrary, possibly more "exotic" Wadge classes?)

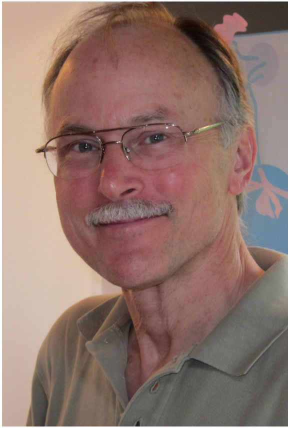
It turns out that Hausdorff operations allow us to do exactly that in a rather elegant way. The first ingredient is the following result, proved by Van Wesep in his Ph.D. thesis:

Theorem (Van Wesep, 1977, for $Z = \omega^\omega$)

The following are equivalent:

- $\mathbf{\Gamma}$ *is a non-selfdual Wadge class in* Z
- *There exists* $D \subseteq \mathcal{P}(\omega)$ *such that* $\mathbf{\Gamma} = \mathbf{\Gamma}_D(Z)$

Robert Van Wesep: medical scientist, mathematician, poet



Plus Ultra

The whole world having been
into its ultrapower injected
The latter being founded well,
if all goes as expected
The sets whose images contain
the point of criticality
Return an ultrafilter with
a dividend: normality!

Relativization: the crucial lemma

The second ingredient is the following "Relativization Lemma". (Similar result have appeared in work of van Engelen, and even earlier in work of Louveau and Saint-Raymond.)
It is hard to understate how much confusion and ugliness was cleared up by this lemma...

Lemma

Let Z and W be arbitrary topological spaces, and let $D \subseteq \mathcal{P}(\omega)$.

- *Assume that $W \subseteq Z$. Then $A \in \mathbf{\Gamma}_D(W)$ iff there exists $\widetilde{A} \in \mathbf{\Gamma}_D(Z)$ such that $A = \widetilde{A} \cap W$*
- *If $f : Z \longrightarrow W$ is continuous and $B \in \mathbf{\Gamma}_D(W)$ then $f^{-1}[B] \in \mathbf{\Gamma}_D(Z)$*
- *If $h : Z \longrightarrow W$ is a homeomorphism then $A \in \mathbf{\Gamma}_D(Z)$ iff $h[A] \in \mathbf{\Gamma}_D(W)$*

Reasonably closed Wadge classes

Given $i \in 2$, set:

$$Q_i = \{x \in 2^\omega : x(n) = i \text{ for all but finitely many } n \in \omega\}$$

Notice that every element of $2^\omega \setminus (Q_0 \cup Q_1)$ is obtained by alternating finite blocks of zeros and finite blocks of ones.

Define the function $\phi : 2^\omega \setminus (Q_0 \cup Q_1) \longrightarrow 2^\omega$ by setting

$$\phi(x)(n) = \begin{cases} 0 & \text{if the } n^{\text{th}} \text{ block of zeros of } x \text{ has even length} \\ 1 & \text{otherwise} \end{cases}$$

where we start counting with the 0^{th} block of zeros. It is easy to check that ϕ is continuous.

Definition (Steel, 1980)

Let $\boldsymbol{\Gamma}$ be a Wadge class in 2^ω. We will say that $\boldsymbol{\Gamma}$ is *reasonably closed* if $\phi^{-1}[A] \cup Q_0 \in \boldsymbol{\Gamma}$ for every $A \in \boldsymbol{\Gamma}$.

Why would anybody need that?

Lemma (Harrington)

Let $\mathbf{\Gamma} = [B]$ be a reasonably closed Wadge class in 2^ω. If $A \leq B$ then this is witnessed by an injective function.

The above lemma will be useful to us because every injective continuous function $f : 2^\omega \longrightarrow 2^\omega$ is an embedding.

Proof.

Let $A^* = \phi^{-1}[A] \cup Q_0$. Since $\mathbf{\Gamma}$ is reasonably closed, we can fix $\sigma : 2^{<\omega} \longrightarrow 2^{<\omega}$ such that $f_\sigma : 2^\omega \longrightarrow 2^\omega$ witnesses $A^* \leq B$. We will construct $\tau : 2^{<\omega} \longrightarrow 2^{<\omega}$ such that $f_\tau : 2^\omega \longrightarrow 2^\omega$ witnesses $A \leq A^*$ and $f_\sigma \circ f_\tau$ is injective.
Make sure that

1. $\tau(s)$ always ends with a 1
2. There are exactly $|s|$ blocks of zeros in $\tau(s)$
3. $s(n)$ is the parity of the n^{th} block of zeros in $\tau(s)$

Begin by setting $\tau(\varnothing) = \langle 1 \rangle$.
Given $s \in 2^{<\omega}$, notice that $\tau(s)^\frown\vec{0} \in A^*$ and $\tau(s)^\frown\vec{1} \notin A^*$.
Since f_σ witnesses that $A^* \leq B$, we must have $f_\sigma(\tau(s)^\frown\vec{0}) \in B$ and $f_\sigma(\tau(s)^\frown\vec{1}) \notin B$. Therefore, we can find $k \in \omega$ such that

$$\sigma(\tau(s)^\frown 0^k) \neq \sigma(\tau(s)^\frown 1^k)$$

Now simply pick $\tau(s^\frown i) \supseteq \tau(s)^\frown i^k$ for $i = 0, 1$ satisfying conditions (1), (2) and (3).
To check that f_τ has the desired properties, observe that

- $\mathrm{ran}(f_\tau) \subseteq 2^\omega \setminus (Q_0 \cup Q_1)$ (By conditions 1 and 2)
- $\phi(f_\tau(x)) = x$ for every $x \in 2^\omega$ (By conditions 1 and 3)

□

Our main tool: Steel's theorem

Given a Wadge class $\mathbf{\Gamma}$ in 2^ω and $X \subseteq 2^\omega$, we will say that X is *everywhere properly* $\mathbf{\Gamma}$ if $X \cap [s] \in \mathbf{\Gamma} \setminus \check{\mathbf{\Gamma}}$ for every $s \in 2^{<\omega}$.

Theorem (Steel, 1980)

Let $\mathbf{\Gamma}$ be a reasonably closed Wadge class in 2^ω. Assume that X and Y are subsets of 2^ω that satisfy the following:

- *X and Y are everywhere properly $\mathbf{\Gamma}$*
- *X and Y are either both meager or both comeager*

Then there exists a homeomorphism $h : 2^\omega \longrightarrow 2^\omega$ such that $h[X] = Y$.

Proof.

Without loss of generality, fix closed nowhere dense subsets X_n and Y_n of 2^ω for $n \in \omega$ such that $X \subset \bigcup_{n\in\omega} X_n$ and $Y \subset \bigcup_{n\in\omega} Y_n$. We will combine Harrington's Lemma with Knaster-Reichbach systems. (To be continued...)

Thank you for your attention

and have a good evening!